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The Fundamental Representation of a Strongly Regular Baer Semigroup

K. S. S. NAMBOORIPAD

Department of Mathematics, University of Kerala, Kariavattom 695 581, India

AND

F. J. PASTIJN

Dienst Hogere Meetkunde, Rijksuniversiteit te Gent, Krijgslaan 281, B-9000 Gent, Belgium

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A bounded partially ordered set I is a complemented modular lattice if and only if the semigroup $B(I)$ of all strongly range-closed residuated transformations of I is a regular semigroup coordinatizing I. If S is any strongly regular Baer semigroup coordinatizing I, then the Janowitz representation of S maps S homomorphically onto a full regular subsemigroup of $B(I)$. It is shown that the Janowitz representation of S is equivalent to Hall's (or Grillet's) fundamental representation.

In this paper we use the notation and the terminology of [2, 4, 5, 14]. We begin by recalling some definitions of [5] (see also [13]) regarding cross-connections.

If P is a partially ordered set and $x \in P$, then $P(x)$ denotes the principal order ideal of P generated by x. Let P and Q be partially ordered sets. An order-preserving mapping $f: P \to Q$ is said to be *normal* if $\operatorname{im} f = Q(a)$ for some $a \in Q$, and for every $x \in P$ there exists a $z \leqslant x$ such that $f \mid P(z)$ is an isomorphism of $P(z)$ onto $Q(xf)$. In particular, if f is normal, then there exists at least one element $b \in P$ such that f is an isomorphism of $P(b)$ onto $Q(a) = \operatorname{im} f$. We denote by $M(f)$ the set of all elements $b \in P$ with this property.

If P, Q, and R are partially ordered sets, and if $f: P \to Q$ and $g: Q \to R$ are normal mappings, then fg is a normal mapping. Consequently, the set $S(P)$ of all normal transformations on a partially ordered set P is a semigroup under composition. Elements of $S(P)$ will be written as right operators, and

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the elements of its left-right dual $S^{\mathrm{op}}(P)$ will be written as left operators. Idempotents of $S(P)$ are called *normal retractions*. A principal ideal $P(a)$ of P is called a *normal retract* if $P(a) = \operatorname{im} e$ for some normal retraction $e \in S(P)$. If every principal ideal of P is a normal retract, then P is called a *regular partially ordered set* [4, 5, 13].

An equivalence relation ρ on a poset P is said to be *normal* if there exists a normal mapping $f \in S(P)$ such that $\ker f = ff^{-1} = \rho$. The poset (under the reverse of inclusion) of all normal equivalences on P will be denoted by P°. Recall from [5] that when P is regular, then so is P°.

If f and g are normal mappings, with $\operatorname{dom} f = \operatorname{dom} g = P$, $\ker f = \ker g$, then it is easy to see that $M(f) = M(g)$. So with each $\rho \in P^{\circ}$ we may associate the subset $M(\rho)$ defined by $M(\rho) = M(f)$, where f is any normal mapping with $\ker f = \rho$. Remark that $a \in M(\rho)$ if and only if $P(a)$ intersects every ρ-class in exactly one element. If this is the case, then $P(a) \cap \rho(x)$ contains a single element which is minimal in its ρ-class, and the mapping $\varepsilon_P(\rho, a)$ which sends each $x \in P$ to the unique element in $P(a) \cap \rho(x)$ is a normal retraction with $\ker \varepsilon_P(\rho, a) = \rho$ and $\operatorname{im} \varepsilon_P(\rho, a) = P(a)$. $\varepsilon_P(\rho, a)$ is called the *projection along ρ upon $P(a)$*.

If f is a normal mapping, with $\operatorname{dom} f = P$, and $\ker f = \rho$, then for any $a \in M(f) = M(\rho)$, one can write $f = \varepsilon_P(\rho, a)\alpha$, where $\alpha = f \mid P(a)$ is an isomorphism of $P(a)$ onto $\operatorname{im} f$. This is called a *normal factorization* of f. Since for every normal mapping f we have $M(f) \neq \square$, every normal mapping has at least one normal factorization, and there is a bijection between the set $M(f)$ and the set of normal factorizations of f.

A regular semigroup S is said to be *fundamental* if the identity congruence is the only idempotent-separating congruence on S. If P is a regular partially ordered set, then $S(P)$ is a fundamental regular semigroup such that $S(P)/\mathscr{L} \approx P$ and $S(P)/\mathscr{R} \approx P^{\circ} = \{\ker f \mid f \in S(P)\}$. Thus, $S(P)$ is a regular semigroup that coordinatizes P [5, 16].

PROPOSITION 1. *Let I and Λ be regular partially ordered sets and let $f: I \to \Lambda$ be a normal mapping. For $\sigma \in \Lambda^{\circ}$, define*

$$f^{\circ}(\sigma) = \ker(f\varepsilon_{\Lambda}(\sigma, u)) = \sigma f^{-1} \tag{1}$$

where $u \in M(\sigma)$. Then $f^{\circ}: \Lambda^{\circ} \to I^{\circ}$ is a normal mapping such that

$$\begin{aligned} &\operatorname{im} f^{\circ} = I^{\circ}(\ker f) \\ &M(f^{\circ}) = \{\rho \in \Lambda^{\circ} \mid b \in M(\rho)\}, \qquad \textit{where} \quad \operatorname{im} f = \Lambda(b). \end{aligned} \tag{2}$$

If P, Q and R are regular partially ordered sets, and if $f: P \to Q$, and $g: Q \to R$ are normal mappings, then

$$(fg)^{\circ} = f^{\circ} g^{\circ}.$$

Proof. Let I and Λ be regular partially ordered sets, and suppose that $f: I \to \Lambda$ is a normal mapping. The mapping $f^\circ: \Lambda^\circ \to I^\circ$ which is given by (1) is order-preserving, and clearly $\operatorname{im} f^\circ \subseteq I^\circ(\ker f)$.

If $\operatorname{im} f = \Lambda(b)$, then we can choose $\rho \in \Lambda^\circ$ such that $b \in M(\rho)$, since Λ is a regular partially ordered set. We can put $f = f\varepsilon_\Lambda(\rho, b)$. Also, if $a \in M(\ker f)$, then $f = \varepsilon_I(\ker f, a)\alpha$, where $\alpha = f \mid I(a)$ maps $I(a)$ isomorphically onto $\Lambda(b)$.

Let $\sigma \in \Lambda^\circ$ and $c \in M(\sigma)$. Since $\varepsilon_\Lambda(\rho, b)$ and $\varepsilon_\Lambda(\sigma, c)$ are idempotents of the regular semigroup $S(\Lambda)$, the sandwich set $S(\varepsilon_\Lambda(\rho, b), \varepsilon_\Lambda(\sigma, c))$ is nonempty. Let $\varepsilon_\Lambda(\sigma', b')$ be any element of this sandwich set. Then $\sigma' \supseteq \sigma$ and $b' \leqslant b$. We put $b'\alpha^{-1} = a' \in I(a)$. If $\tau = f^\circ(\sigma)$, we have

$$\begin{aligned}(u, v) \in \tau &\Leftrightarrow (uf)\, \varepsilon_\Lambda(\sigma, c) = (vf)\, \varepsilon_\Lambda(\sigma, c)\\ &\Leftrightarrow (uf)\, \varepsilon_\Lambda(\sigma', b') = (vf)\, \varepsilon_\Lambda(\sigma', b')\\ &\Leftrightarrow (u, v) \in f^\circ(\sigma'),\end{aligned}$$

and thus $\tau = f^\circ(\sigma')$. Let $\sigma'' \supseteq \sigma'$. Then there exists $b'' \in M(\sigma'')$ such that $b'' \leqslant b' \leqslant b$. Since $f^\circ(\sigma'') = \ker(f\varepsilon_\Lambda(\sigma'', b''))$, $f\varepsilon_\Lambda(\sigma'', b'')\,\alpha^{-1}$ is a normal retraction of I such that $\operatorname{im}(f\varepsilon_\Lambda(\sigma'', b'')\,\alpha^{-1}) = I(b''\alpha^{-1})$ and $\ker(f\varepsilon_\Lambda(\sigma'', b'')\,\alpha^{-1}) = f^\circ(\sigma'')$. Thus for all $\sigma'' \supseteq \sigma'$ and $b'' \in M(\sigma'')$, with $b'' \leqslant b'$, one has

$$f\varepsilon_\Lambda(\sigma'', b'') = \varepsilon_I(f^\circ(\sigma''), b''\alpha^{-1})\,\alpha.$$

In particular we obtain

$$f\varepsilon_\Lambda(\sigma', b') = \varepsilon_I(f^\circ(\sigma'), a')\,\alpha = \varepsilon_I(\tau, a')\,\alpha.$$

We now show that f° is an isomorphism of $\Lambda^\circ(\sigma')$ onto $I^\circ(\tau)$. Let $\tau'' \supseteq \tau$, and let $\sigma'' = \ker h$, where

$$h = \varepsilon_\Lambda(\sigma', b')\,\alpha^{-1}\varepsilon_I(\tau'', a''),$$

with

$$a'' \in M(\tau'') \text{ and } a'' \leqslant a' = b'\alpha^{-1} \leqslant a.$$

Then $\sigma'' \in \Lambda^\circ$ and $(x, y) \in f^\circ(\sigma'')$ if and only if $xfh = yfh$. But

$$\begin{aligned}xfh &= xf\varepsilon_\Lambda(\sigma', b')\,\alpha^{-1}\varepsilon_I(\tau'', a'')\\ &= x\varepsilon_I(\tau, a')\,\alpha\alpha^{-1}\varepsilon_I(\tau'', a'')\\ &= x\varepsilon_I(\tau, a')\,\varepsilon_I(\tau'', a'')\\ &= x\varepsilon_I(\tau'', a'').\end{aligned}$$

Thus $xfh = yfh$ if and only if $(x, y) \in \tau''$, in other words, $f^\circ(\sigma'') = \tau''$. This proves that f° maps $\Lambda^\circ(\sigma')$ onto $I^\circ(\tau)$.

It is clear that $f^\circ \mid \Lambda^\circ(\sigma')$ preserves order. Assume that $\sigma_i \supseteq \sigma'$, $f^\circ(\sigma_i) = \tau_i \supseteq \tau$, $i = 1, 2$, and $\tau_1 \supseteq \tau_2$. Then $(u, v) \in \sigma_i$ if and only if $(u\varepsilon_\Lambda(\sigma', b'), v\varepsilon_\Lambda(\sigma', b')) \in \sigma_i$, $i = 1, 2$. Thus, putting $x = u\varepsilon_\Lambda(\sigma', b')\,\alpha^{-1}$, $y = v\varepsilon_\Lambda(\sigma', b')\,\alpha^{-1}$, then

$$\begin{aligned}(u, v) \in \sigma_2 &\Rightarrow (xf, yf) \in \sigma_2\\ &\Rightarrow (x, y) \in f^\circ(\sigma_2) = \tau_2\\ &\Rightarrow (x, y) \in f^\circ(\sigma_1) = \tau_1\\ &\Rightarrow (xf, yf) \in \sigma_1\\ &\Rightarrow (u, v) \in \sigma_1,\end{aligned}$$

from which $\sigma_2 \subseteq \sigma_1$. This proves that $f^\circ \mid \Lambda^\circ(\sigma')$ is an order isomorphism onto $I^\circ(\tau)$.

If in the above reasoning we put $\sigma = \rho$, then we obtain $\sigma' = \rho$ and $\tau = \ker f$. From this we infer $\operatorname{im} f^\circ = I^\circ(\ker f)$, and we proved that f° is normal. From the above also follows that $M(f^\circ) \supseteq \{\rho \in \Lambda^\circ \mid b \in M(\rho)\}$. Let us now suppose that $\sigma \in M(f^\circ)$, i.e., $f^\circ \mid \Lambda^\circ(\sigma)$ is an isomorphism of $\Lambda^\circ(\sigma)$ onto $I^\circ(\ker f)$. The above reasoning shows that there exists a $\sigma' \supseteq \sigma$, $b' \in M(\sigma')$, $b' \leqslant b$, such that $f^\circ \mid \Lambda^\circ(\sigma')$ is an isomorphism of $\Lambda^\circ(\sigma')$ onto $I^\circ(\ker f)$. From this we have $\sigma = \sigma'$, whence $M(\sigma)$ contains an element b', with $b' \leqslant b$. If $b' \neq b$, then $(b'\alpha^{-1}, b\alpha^{-1}) \in f^\circ(\sigma)$ and $(b'\alpha^{-1}, b\alpha^{-1}) \notin \ker f$. This is of course impossible since $f^\circ(\sigma) = \ker f$. Thus $b \in M(\sigma)$, and (2) follows.

We use the result proved above to reformulate the definition of cross-connections [5, 13].

Suppose that I and Λ are regular partially ordered sets and $\Gamma\colon \Lambda \to I^\circ$, $\Delta\colon I \to \Lambda^\circ$ are order preserving mappings. We say that $(f, g) \in S^{\mathrm{op}}(I) \times S(\Lambda)$ is compatible with (Γ, Δ) if the following conditions hold:

(C1) $\operatorname{im} f = I(x)$, $\operatorname{im} g = \Lambda(y) \Rightarrow \ker f = \Gamma(y)$, $\ker g = \Delta(x)$,

(C2) the following diagrams commute:

$$\begin{array}{ccc} I & \xrightarrow{\Delta} & \Lambda^\circ \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle g^\circ} \\ I & \xrightarrow{\Delta} & \Lambda^\circ \end{array} \qquad\qquad \begin{array}{ccc} \Lambda & \xrightarrow{\Gamma} & I^\circ \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle f^\circ} \\ \Lambda & \xrightarrow{\Gamma} & I^\circ \end{array}$$

(D1) (D2)

We observe that if I and Λ have identity elements, and if Γ and Δ preserve identity elements, then (C2) implies (C1). In this case the identity relations on I and Λ are the identity elements of I° and Λ°, respectively.

In view of Proposition 1, the definition of cross-connections (see [5, 13, 16]) is equivalent to the following.

THEOREM 2. *Let I, Λ be regular partially ordered sets and let $\Gamma: \Lambda \to I^{\circ}$, $\Delta: I \to \Lambda^{\circ}$ be order preserving mappings. Then $[I, \Lambda; \Gamma, \Delta]$ is a cross-connection if and only if the following conditions are satisfied*:

(Cr1) $x \in M(\Gamma(y)) \Leftrightarrow y \in M(\Delta(x))$, $x \in I$, $y \in \Lambda$,

(Cr2) *if* $x \in M(\Gamma(y))$, *then the pair*

$$(\varepsilon_I(\Gamma(y), x), \varepsilon_\Lambda(\Delta(x), y))$$

is compatible with (Γ, Δ).

Remark that the fundamental regular semigroup $U = U(I, \Lambda; \Gamma, \Delta)$ considered in [13] consists of all the pairs (f, g) that are compatible with (Γ, Δ).

Recall from [16] that an isomorphism between two cross-connections $[I, \Lambda; \Gamma, \Delta]$ and $[I', \Lambda'; \Gamma', \Delta']$ is a pair of order isomorphisms $\alpha: I \to I'$, $\beta: \Lambda \to \Lambda'$, such that

$$\Delta = \alpha \circ \Delta' \circ \beta^{\circ}, \qquad \Gamma = \beta \circ \Gamma' \circ \alpha^{\circ},$$

where α°: $I'^{\circ} \to I^{\circ}$ and $\beta^{\circ}: \Lambda'^{\circ} \to \Lambda^{\circ}$ are defined as in Proposition 1. For any fundamental regular semigroup S, define $I_S = S/\mathscr{R}$, $\Lambda_S = S/\mathscr{L}$,

$$\lambda_e: I_S \to I_S, R_g \to R_{eg}, \qquad e, g \in E(S),$$

$$\rho_e: \Lambda_S \to \Lambda_S, L_g \to L_{ge}, \qquad e, g \in E(S),$$

$$\Gamma_S(L_e) = \ker \lambda_e, \qquad \Delta_S(R_e) = \ker \rho_e, \qquad e \in E(S).$$

Then $[I_S, \Lambda_S; \Gamma_S, \Delta_S]$ is a cross-connection, and S is isomorphic to a full regular subsemigroup of $U(I, \Lambda; \Gamma, \Delta)$ if and only if the cross-connection $[I_S, \Lambda_S; \Gamma_S, \Delta_S]$ induced by S is isomorphic to the cross-connection $[I, \Lambda; \Gamma, \Delta]$ (see [5, 16]).

An order preserving mapping $f: P \to Q$ of a poset P into a poset Q is said to be *residuated* if there exists an order preserving mapping $f^+: Q \to P$ such that $yf^+f \leqslant y$ and $x \geqslant xff^+$ for all $x \in P$ and $y \in Q$. The mapping f^+ is called the *residual* of f [2]. It is easy to see that if the residual f^+ of f exists, then it is unique. Further, if f and g are residuated mappings, and if fg exists, then it is a residuated mapping and its residual is g^+f^+. It follows that the set $\operatorname{Res} P$ of all residuated transformations of P is a semigroup under the composition of transformations, and that $f \to f^+$ is a dual isomorphism of

$\operatorname{Res} P$ onto the semigroup $\operatorname{Res}^+ P$ of all residuals of elements of $\operatorname{Res} P$. It is clear that each $f^+ \in \operatorname{Res}^+ P$ is a residuated transformation of P^{op} with residual $f \in \operatorname{Res} P$. Thus there is a natural isomorphism between $\operatorname{Res}^+ P$ and $\operatorname{Res} P^{\mathrm{op}}$ [8]. For convenience we shall henceforth identify $\operatorname{Res}^+ P$ with $\operatorname{Res} P^{\mathrm{op}}$ and regard f^+ as a residuated transformation on P^{op} whenever $f \in \operatorname{Res} P$. We shall also write elements of $\operatorname{Res} P$ [$\operatorname{Res} P^{\mathrm{op}}$] as right [left] operators. It follows that $\operatorname{Res} P \to \operatorname{Res} P^{\mathrm{op}}$, $f \to f^+$ is an isomorphism. For all $f \in \operatorname{Res} P$ and $x \in P$ we thus have

$$(f^+x)f \leqslant x \leqslant f^+(xf). \tag{3}$$

We say that $f \in \operatorname{Res} P$ is *totally range closed* if f maps principal ideals onto principal ideals [2, 8]. Observe that a residuated transformation that is also normal must be totally range closed. Further, $f \in \operatorname{Res} P$ is said to be *strongly range closed* if f and f^+ are totally range closed transformations of P and P^{op} respectively. It is clear that the set $B(P)$ of all strongly range closed transformations of P is a subsemigroup of $\operatorname{Res} P$, and $f \to f^+$ is an isomorphism of $B(P)$ onto $B(P^{\mathrm{op}})$. If $f \in \operatorname{Res} P$, and if both f and f^+ are normal, then we shall say that f is *binormal*; if this is the case, then $f \in B(P)$. Again, the set of all binormal transformations of P forms a subsemigroup of $B(P)$.

PROPOSITION 3. *Let L be a complemented modular lattice, let $a \in L$ and let a' be a complement of a in L. Then*

$$(a; a'): L \to L, x \to (x \vee a) \wedge a' \tag{4}$$

is a binormal idempotent transformation such that

$$(a; a')^+ : L^{\mathrm{op}} \to L^{\mathrm{op}}, \qquad y \to (y \wedge a') \vee a \tag{5}$$

is the residual of $(a; a')$. Further,

$$\begin{aligned} \ker(a; a') &= \Delta(a) = \{(x, y) \mid x \vee a = y \vee a\} \\ \ker(a; a')^+ &= \Gamma(a') = \{(x, y) \mid x \wedge a' = y \wedge a'\} \end{aligned} \tag{6}$$

and

$$M(\Gamma(a)) = M(\Delta(a)) = \{a' \mid a' \textit{ is a complement of } a \textit{ in } L\}. \tag{7}$$

Conversely, if e is any strongly range closed idempotent transformation of L, then e is binormal and $e = (a; a')$ for some $a \in L$ and some complement a' of a in L.

Proof. It is noted in [17] that $(a; a')$ is a normal transformation. We nevertheless provide a proof for the sake of completeness.

For any $x \in L$, we have

$$\begin{aligned}(a; a')^+(x(a; a')) &= (((x \vee a) \wedge a') \wedge a') \vee a \\ &= ((x \vee a) \wedge a') \vee a \\ &= (x \vee a) \wedge (a' \vee a) \qquad \text{(by modularity)} \\ &= x \vee a,\end{aligned}$$

and

$$((a; a')^+ x)(a; a') = x \wedge a'.$$

Since $(a; a')$ and $(a; a')^+$ are order preserving, it follows that $(a; a') \in \operatorname{Res} L$ and $(a; a')^+$ is the residual of $(a; a')$.

To prove that $\phi = (a; a')$ is normal, consider $x \in L$ and let $z = (a \wedge x)' \wedge x$, where $(a \wedge x)'$ is a complement of $a \wedge x$. Then $z \vee a \geqslant z \vee (x \wedge a) = x \wedge ((x \wedge a)' \vee (a \wedge x)) = x$ by modularity. So $z \vee a \geqslant x \vee a$. Clearly $z \vee a \leqslant x \vee a$, whence $x \vee a = z \vee a$. Consequently $x\phi = z\phi$. Suppose now that $z_1, z_2 \in L(z)$. Then

$$\begin{aligned}z_1\phi \leqslant z_2\phi &\Rightarrow z_1\phi \vee a \leqslant z_2\phi \vee a \\ &\Rightarrow z_1 \vee a \leqslant z_2 \vee a \qquad \text{(by modularity)} \\ &\Rightarrow (z_1 \vee a) \wedge z \leqslant (z_2 \vee a) \wedge z \\ &\Rightarrow z_1 \leqslant z_2 \qquad \text{(by modularity and since } a \wedge z = 0).\end{aligned}$$

It follows that ϕ is an order-isomorphism of $L(z)$ onto $L(z\phi) = L(x\phi)$. Thus $(a; a')$ is normal. By duality it follows that $(a; a')^+$ is a normal transformation of L^{op}. Thus $(a; a')$ is binormal. Also

$$(z_1, z_2) \in \ker(a; a') \Leftrightarrow z_1 \vee a = z_2 \vee a$$

as before, and so $\ker(a; a') = \Delta(a)$ is given by (6). Dually, $\ker(a; a')^+ = \Gamma(a')$ is given by (6).

Since for every complement a' of a, $(a; a')$ is a normal idempotent with $\ker(a; a') = \Delta(a)$ and $\operatorname{im}(a; a') = L(a')$, it follows that every complement of a belongs to $M(\Delta(a))$. If $t \in M(\Delta(a))$, then $\phi = (a; a')$ is an isomorphism of $L(t)$ onto $L(a')$. In particular $(t \vee a) \wedge a' = t\phi = a'$, and so $a' \leqslant t \vee a$. Thus $1 = a \vee a' \leqslant t \vee a \leqslant 1$. Further $t \wedge a \in L(t)$ and $(t \wedge a)\phi \leqslant a\phi = 0$. Hence $0 = (t \wedge a)\phi$. Since 0, $t \wedge a \in L(t)$ and since ϕ is an isomorphism when restricted to $L(t)$, we conclude that $t \wedge a = 0$. Thus t is a complement of a, and (7) holds.

If e is a strongly range closed idempotent transformation of L, then it follows from Theorem 13.4 of [2] that $e = (e^+0; 1e)$, and so by the above proof e is binormal.

Our result above shows that if L is a complemented modular lattice, then for every $a \in L$ there exist binormal idempotent transformations e and f such that $1e = a = f^+0$. The following shows that this property characterizes complemented modular lattices.

THEOREM 4. *For a poset I with* 0 *and* 1 *the following are equivalent*:

(i) *I is a complemented modular lattice*,

(ii) *for each $a \in I$ there exist binormal idempotent transformations e and f such that* $1e = a = f^+0$,

(iii) *$B(I)$ is a regular semigroup, and*

$$B(I)/\mathscr{L} \to I, \quad L_g \to 1g$$
$$B(I^{\text{op}})/\mathscr{R} \to I, \quad R_h \to h0$$

are isomorphisms.

Proof. (i) $\Rightarrow$ (ii) follows from Proposition 3.

(ii) $\Rightarrow$ (iii). From Theorem 13.2 of [2] follows that I is a bounded lattice. From Theorem 14.5 of [2] and the corollary preceding it then follows that $B(I)$ is a regular semigroup. In view of (ii) we may conclude that $B(I)$ coordinatizes $B(I)$.

(iii) $\Rightarrow$ (i) follows from Theorem 14.6 of [2].

Let L be a lattice with 0 and 1. For each $a \in L$ the relation

$$\Gamma(a) = \{(x, y) \mid x \wedge a = y \wedge a\} \tag{8}$$

is an equivalence relation on L, and the mapping

$$\Gamma: L \to \text{Eq}(L), \qquad a \to \Gamma(a),$$

is an order preserving embedding of L into the poset $\text{Eq}(L)$ of all equivalence relations on L ordered under the reverse of inclusion. Note that $\Gamma(a) = \ker e_a$, where $e_a: L \to L$, $x \to x \wedge a$ is a normal retraction of L. Hence $\Gamma(a) \in L^\circ$ for all $a \in L$, and

$$\Gamma: L \to L^\circ, \qquad a \to \Gamma(a) \tag{9}$$

is an order preserving embedding of L into L°. Proposition 3 shows that if L is a complemented modular lattice, Γ is an order preserving embedding of L into $(L^{\text{op}})^\circ$ also. Dually,

$$\Delta(a) = \{(x, y) \mid x \vee a = y \vee a\} \tag{10}$$

is a normal equivalence on L^{op}, and

$$\Delta: L^{\mathrm{op}} \to (L^{\mathrm{op}})^{\circ}, \qquad a \to \Delta(a) \tag{11}$$

is an order preserving embedding of L^{op} into $(L^{\mathrm{op}})^{\circ}$. Again by Proposition 3 we see that if L is a complemented modular lattice, then Δ is also an order preserving embedding of L^{op} into L°.

It may be noted that if L, Γ and Δ are given by the above, then $[L, L; \Gamma, \Gamma]$ and $[L^{\mathrm{op}}, L^{\mathrm{op}}; \Delta, \Delta]$ are the cross-connections induced by the semilattices L and L^{op}, respectively.

PROPOSITION 5. *Let L be a complemented modular lattice and let Γ and Δ be given by* (8), (9), (10) *and* (11). *For an order preserving transformation $f: L \to L$ the following statements are equivalent*:

(i) *f is normal and* $\ker f = \Delta(a)$ *for some $a \in L$,*

(ii) *f is strongly range closed,*

(iii) *f is binormal,*

(iv) *there exists a normal transformation $g: L^{\mathrm{op}} \to L^{\mathrm{op}}$ such that (g, f) is compatible with (Γ, Δ).*

Moreover, if f satisfies these conditions, then f^{+} is the unique normal transformation of L^{op} such that (f^{+}, f) is compatible with (Γ, Δ).

Proof. (i) $\Rightarrow$ (ii). Since f is normal, its image is a principal ideal of L. If $1f = b$, then $\operatorname{im} f = L(b)$. Thus, for all $y \in L$, there exists $x \in L$ such that $xf = y \wedge b$. Since $\ker f = \Delta(a)$ we have $xf = zf$ if and only if if $x \vee a = z \vee a$. Thus $\bigvee \{z \in L \mid zf = y \wedge b\}$ exists, and is equal to $x \vee a$ for some $x \in L$ with $xf = y \wedge b$. For $y \in L^{\mathrm{op}}$, define $f^{+}y$ by

$$f^{+}y = \bigvee \{z \in L \mid zf = y \wedge b\}.$$

Then $f^{+}: L^{\mathrm{op}} \to L^{\mathrm{op}}$ is well-defined. Further, we have

$$(f^{+}y)f = y \wedge b, \qquad f^{+}(xf) = x \vee a$$

for all $x, y \in L$. If $y \leqslant y'$, then there exist x, $x' \in L$ with $x \leqslant x'$ such that $xf = y \wedge b$, $x'f = y' \wedge b$ since f is a normal transformation of L. Thus $f^{+}y = x \vee a \leqslant x' \vee a = f^{+}y'$. Thus f^{+} is order preserving, and so f is residuated, f^{+} being its residual. Since f is normal, it is also totally range closed. Obviously $\operatorname{im} f^{+}$ is the principal ideal of L^{op} which is generated by a. Hence by Corollary 2 to Theorem 13.7 of [2], f is strongly range closed.

(ii) $\Rightarrow$ (iii). If f is strongly range closed, then by Theorem 14.5 and Corollary 2 to Theorem 13.7 of [2] there exists a strongly range closed transformation $g: L \to L$ such that $f = fgf$ and $g = gfg$. Hence $e = fg$ is a strongly range closed idempotent transformation, and $e = (a'; a)$ with $a = 1g$ and a' a complement of a in L by Proposition 3. Now $\alpha = f \mid L(a)$ is an isomorphism of $L(a)$ onto $L(b) = \operatorname{im} f$, and $f = e\alpha$. By Proposition 3, e is normal. Hence f is normal. Similarly f^+ is strongly range closed, and so f^+ is also normal. We conclude that f is binormal.

(iii) $\Rightarrow$ (iv). We show that (f^+, f) is compatible with (Γ, Δ), that is, for all $x \in L$,

$$\Gamma(xf) = (f^+)^\circ(\Gamma(x)) = (f^+)^{-1}(\Gamma(x)),$$

and

$$\Delta(f^+x) = (\Delta(x))\, f^\circ = \Delta(x)\, f^{-1}.$$

Suppose that $(u, v) \in \Gamma(xf)$. Then $u \wedge xf = v \wedge xf$. Now f^+ preserves the meet operation (see [2, Exercise 4.2], or [8]). Hence $f^+(u \wedge xf) = f^+u \wedge f^+(xf)$. By Theorem 13.1* of [2], $f^+(xf) = x \vee f^+0$. Hence

$$\begin{aligned} f^+(u \wedge xf) &= f^+u \wedge (x \vee f^+0) \\ &= (f^+u \wedge x) \vee f^+0 \end{aligned}$$

by modularity. Now again by modularity

$$\begin{aligned} f^+u \wedge x &= (f^+u \wedge x) \vee (f^+0 \wedge x) \\ &= ((f^+u \wedge x) \vee f^+0) \wedge x \\ &= f^+(u \wedge xf) \wedge x \end{aligned}$$

and similarly

$$f^+v \wedge x = f^+(v \wedge xf) \wedge x.$$

From $u \wedge xf = v \wedge xf$ then follows $f^+u \wedge x = f^+v \wedge x$. Hence $(f^+u, f^+v) \in \Gamma(x)$, and thus $(u, v) \in (f^+)^{-1}(\Gamma(x))$.

If $(u, v) \in (f^+)^{-1}(\Gamma(x))$, then $f^+u \wedge x = f^+v \wedge x$. The binormal transformation f is in particular strongly range closed, so by Theorem 13.6 of [2],

$$u \wedge xf = (f^+u \wedge x)f = (f^+v \wedge x)f = v \wedge xf,$$

that is, $(u, v) \in \Gamma(xf)$. We conclude $\Gamma(xf) = (f^+)^{-1}(\Gamma(x))$. The equality $\Delta(f^+x) = \Delta(x)f^{-1}$ is proved dually.

(iv) $\Rightarrow$ (i). If $g: L^{\mathrm{op}} \to L^{\mathrm{op}}$ is a normal transformation such that (g, f) is compatible with (Γ, Δ), then

$$\Gamma(1f) = g^{-1}(\Gamma(1)) = \ker g$$

since $\Gamma(1)$ is the equality relation. Hence g satisfies the dual of (i), and from the above proof for (i) $\Rightarrow$ (ii) we now see that g is strongly range closed, and from the proofs for (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) we have that g is binormal and that (g^+, g) is compatible with (Γ, Δ). Hence for all $x \in L$,

$$\Gamma(xg^+) = g^\circ(\Gamma(x)) = \Gamma(xf).$$

Since Γ is injective, $xg^+ = xf$ for all $x \in L$, thus $f = g^+$. Since g is binormal, $f = g^+$ must be normal. Further,

$$\Delta(g0) = \Delta(0)f^{-1} = \ker f$$

since $\Delta(0)$ is the equality relation. Hence (i) holds.

THEOREM 6. *Let L be a lattice with* 0 *and* 1, *and define Γ and Δ by* (8), (9), (10) *and* (11). *Then the following are equivalent*:

(i) *L is a complemented modular lattice*,

(ii) *Δ is an order embedding of L^{op} into L°*,

(iii) *Γ is an order embedding of L into $(L^{\mathrm{op}})^\circ$*,

(iv) *$[L^{\mathrm{op}}, L; \Gamma, \Delta]$ is a cross-connection.*

If these conditions are satisfied, then the fundamental regular semigroup $U = U(L^{\mathrm{op}}, L; \Gamma, \Delta)$ is given by

$$U = \{(f^+, f) \mid f \in B(L)\}. \tag{12}$$

Proof. That (i) implies (ii) and (iii) follows from Proposition 3, and that (iv) implies (ii) and (iii) follows from the definition of a cross-connection. It will be sufficient to prove that (ii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (iv) hold: the proof for (iii) $\Rightarrow$ (i) will follow by duality.

(ii) $\Rightarrow$ (i). Let $a \in L$, and let $a' \in M(\Delta(a))$. Let e be the projection along $\Delta(a)$ upon $L(a')$. Since $\operatorname{im} e = L(a')$ and $\ker e = \Delta(a)$, we may define

$$e^+y = \bigvee \{x \in L \mid xe = y \wedge a'\}, \qquad y \in L,$$

and then

$$(e^+y)e = y \wedge a', \quad e^+(xe) = x \vee a, \qquad x, y \in L,$$

so that e is a residuated transformation and e^+ its residual. Consequently $\operatorname{im} e^+ = L^{\mathrm{op}}(a)$, and from Theorem 13.4 of [2] we then have that $a' = 1e$ and $a = e^+0$ are complements in L. Thus L is complemented. Obviously e is a totally range closed idempotent transformation of L for all $a \in L$, $a' \in M(\Delta(a))$. Therefore by Corollary 1 to Theorem 13.7 of [2], (b, a) is a dual modular pair for all $a, b \in L$. Thus L is a modular lattice.

(i) $\Rightarrow$ (iv). By Proposition 3, $a' \in M(\Gamma(a))$ if and only if a' is a complement of a, and then $a \in M(\Delta(a'))$. Thus the condition (Cr1) of Theorem 2 holds. By Proposition 5, $((a;a')^+, (a;a'))$ is compatible with (Γ, Δ), since $(a;a')$ is binormal. Thus (Cr2) also holds and $[L^{\text{op}}, L; \Gamma, \Delta]$ is a cross-connection.

$U = U(L^{\text{op}}, L; \Gamma, \Delta)$ is the semigroup of all the pairs of normal transformations (g, f), with $g: L^{\text{op}} \to L^{\text{op}}$, $f: L \to L$, that are compatible with (Γ, Δ). Using Proposition 5, we must have (12).

COROLLARY 7. *If the equivalent conditions of Theorem* 6 *are satisfied, then*

$$U(L^{\text{op}}, L; \Gamma, \Delta) \to B(L), \qquad (f^+, f) \to f \tag{13}$$

is an isomorphism.

If L is a complemented modular lattice, let $P(L)$ denote the subsemigroup of $B(L)$ which is generated by the idempotent transformations $(a;a')$, where a and a' are complements in L. Then from [17] we have that $P(L)$ is the idempotent-generated part of $B(L)$ and $E(P(L)) = E(B(L))$ consists of the pairs $(a;a')$, a and a' complementary in L. The biordered set $E(L)$ of idempotents of $B(L)$ (and of $P(L)$) was constructed in [17] in terms of the complemented modular lattice L. Using Theorem C of [13] one can construct yet another isomorphic copy $T(E(L))$ of $B(L) \cong U(L^{\text{op}}, L; \Gamma, \Delta)$.

Recall from [2, 7, 8] that the right annihilator X^r of a subset X of a semigroup S with zero 0 is given by

$$X^r = \{y \in S \mid xy = 0 \text{ for all } x \in X\}.$$

We write x^r instead of $\{x\}^r$. The left annihilators X^l and x^l are defined dually. A regular semigroup S with 0 is called a *strongly regular Baer semigroup* if the set of all principal left ideals of S coincides with the set of left annihilators of the elements of S and if the set of all principal right ideals of S coincides with the set of right annihilators of the elements of S. In other words, there exist surjective mappings

$$\sigma_l: S \to S/\mathscr{L}, \qquad \sigma_r: S \to S/\mathscr{R},$$

such that for all $x \in S$,

$$S(x\sigma_l) = x^l, \qquad (x\sigma_r)S = x^r.$$

From [2] we know that if this is the case, then $S/\mathscr{L}$ and $S/\mathscr{R}$ are dually isomorphic complemented modular lattices. It is easy to verify that if S is a strongly regular Baer semigroup, then every $\mathscr{H}$-coextension of a full regular subsemigroup of S is also a strongly regular Baer semigroup. Further, if L is a complemented modular lattice, then $B(L)$ is a strongly regular Baer semigroup [8].

THEOREM 8. *A regular semigroup S is a strongly regular Baer semigroup if and only if the cross-connection $[I_S, \Lambda_S; \Gamma_S, \Delta_S]$ which is induced by S is isomorphic to the cross-connection $[L^{\mathrm{op}}, L; \Gamma, \Delta]$ for some complemented modular lattice L.*

Proof. Let S be a strongly regular Baer semigroup. Then $L = S/\mathscr{L}$ is a complemented modular lattice. If μ is the greatest idempotent-separating congruence relation on S, then $S/\mu = T$ is a fundamental strongly regular Baer semigroup, and by Theorem 3 of [17] the idempotent-generated part of T is isomorphic to $P(L)$, i.e., the idempotent-generated part of $B(L)$. Consequently the biordered set of idempotents of S is isomorphic to the biordered set $E(L)$ of idempotents of $B(L)$. By Theorems B and C of [13] and by Corollary 7 we then have

$$U(I_S, \Lambda_S; \Gamma_S, \Delta_S) \cong T(E(S)) \cong T(E(L)) \cong B(L) \cong U(L^{\mathrm{op}}, L; \Gamma, \Delta),$$

and so by Theorem 3.2 of [16], the cross-connection $[I_S, \Lambda_S; \Gamma_S, \Delta_S]$ is isomorphic to the cross-connection $[L^{\mathrm{op}}, L; \Gamma, \Delta]$.

Conversely, if $[I_S, \Lambda_S; \Gamma_S, \Delta_S]$ is isomorphic to $[L^{\mathrm{op}}, L; \Gamma, \Delta]$, then by Theorem 3.2 of [16], S/μ is isomorphic to a full regular subsemigroup of $U(L^{\mathrm{op}}, L; \Gamma, \Delta)$ which in its turn is isomorphic to the strongly regular Baer semigroup $B(L)$. It follows that S is also a strongly regular Baer semigroup.

Remark 9. In view of the isomorphism $U(I_S, \Lambda_S; \Gamma_S, \Delta_S) \cong T(E(S))$ for any regular semigroup S, we could have stated the above result equivalently in terms of biordered sets as follows. A regular semigroup S is a strongly regular Baer semigroup if and only if the biordered set $E(S)$ of idempotents of S is isomorphic to the biordered set $E(L)$ of idempotents of $B(L)$ for some complemented modular lattice L.

The following corollary to Theorems 6 and 8 is analogous to Theorem 3 of [17].

COROLLARY 10. *Let L be a complemented modular lattice. The semigroup S is a fundamental strongly regular Baer semigroup coordinatizing L if and only if S is isomorphic to a full regular subsemigroup of $B(L)$, and $B(L)$ is determined up to isomorphism by this property.*

Remark 11. Let L be a complemented modular lattice. It follows from Theorem 1 of [17] that $E(L) = E(B(L))$ is a semilattice if and only if L is uniquely complemented, that is, if and only if L is a Boolean algebra (see also Theorem 18.12 of [2]). In this case the idempotent $(a; a')$ of $B(L)$ is the normal retraction $e_{a'}: L \to L$, $x \to x \wedge a'$ onto the principal ideal $L(a')$ of L. The biordered set $E(L)$ which is induced by $B(L)$ is then isomorphic to the biordered set induced by the meet semilattice $(L, \wedge)$. The strongly regular Baer semigroup $B(L)$ which is isomorphic to $T(E(L))$ is then an isomorphic copy of the Munn semigroup $T(L)$ of the meet semilattice $(L, \wedge)$. In fact, the mapping

$$B(L) \to T(L), \qquad f \to f \mid L((f^+0)')$$

is an isomorphism, where $(f^+0)'$ denotes the unique complement of f^+0 in L. Dually, $B(L)$ is also isomorphic to the Munn semigroup $T^{\mathrm{op}}(L)$ of the join semilattice $(L, \vee)$. In view of this and Theorem 3.2 of [16] we have that the cross-connections $[L, L; \Gamma, \Gamma]$, $[L^{\mathrm{op}}, L; \Gamma, \Delta]$ and $[L^{\mathrm{op}}, L^{\mathrm{op}}; \Delta, \Delta]$ are isomorphic.

If $\phi_i: S \to T_i$, $i = 1, 2$, are two representations of a semigroup S, we say that ϕ_1 is equivalent to ϕ_2 if there exists an isomorphism ϕ: $\operatorname{im} \phi_1 \to \operatorname{im} \phi_2$ such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "\phi_1"] \arrow[dr, "\phi_2"'] & \operatorname{im} \phi_1 \arrow[d, "\phi"] \\ & \operatorname{im} \phi_2 \end{tikzcd}$$

(D3)

Let S be a strongly regular Baer semigroup, let L_l be the poset of principal left ideals of S, and let L_r be the poset of principal right ideals of S. From [2, 8] we know that L_l and L_r are dually isomorphic complemented modular lattices. Further, for each $x \in S$

$$\phi_x: L_l \to L_l, \qquad Se \to ((Sex)^r)^l = Sex$$

is a strongly range-closed transformation on L_l and

$$\phi: S \to B(L_l), \qquad x \to \phi_x \tag{14}$$

is a representation of S [2, 8]. The homomorphism ϕ is called the *right Janowitz representation* of S. For each $x \in S$,

$$\psi_x: L_r \to L_r, \qquad eS \to ((xeS)^r)^l = xeS$$

is a strongly range closed transformation of L_r and

$$\psi: S \to B^{\mathrm{op}}(L_r), \qquad x \to \psi_x \tag{15}$$

is a representation of S [2, 8]. The homomorphism ψ will be called the *left Janowitz representation* of S, and

$$(\psi, \phi): S \to B^{\mathrm{op}}(L_r) \times (L_l), \qquad x \to (\psi_x, \phi_x) \tag{16}$$

the (two-sided) *Janowitz representation of* S.

Let $I_S = S/\mathscr{R}$, $\Lambda_S = S/\mathscr{L}$, and for $x \in S$, consider the transformations

$$\lambda_x : I_S \to I_S, \qquad R_e \to R_{xe},$$
$$\rho_x : \Lambda_S \to \Lambda_S, \qquad L_e \to L_{ex}.$$

Then $\lambda_x \in S^{\mathrm{op}}(I_S)$ and $\rho_x \in S(\Lambda_S)$ for all $x \in S$ [4], and

$$\lambda : S \to S^{\mathrm{op}}(I_S), \qquad x \to \lambda_x, \tag{17}$$

$$\rho : S \to S(\Lambda_S), \qquad x \to \rho_x \tag{18}$$

are representations of S ([4, 6]). It was observed in [5] that

$$(\lambda, \rho): S \to S^{\mathrm{op}}(I_S) \times S(\Lambda_S), \qquad x \to (\lambda_x, \rho_x) \tag{19}$$

maps S onto a full regular subsemigroup of $U(I_S, \Lambda_S; \Gamma_S, \Delta_S)$. We call λ [ρ, (λ, ρ)] the *left* [*right, two-sided*] *Hall representation* of S.

THEOREM 12. *Let S be a strongly regular Baer semigroup. Then the representations ψ, ϕ, (ψ, ϕ), λ, ρ and (λ, ρ) which are given by* (14), (15), (16), (17), (18) *and* (19) *are equivalent and*

$$\ker \lambda = \ker \rho = \ker(\lambda, \rho) = \ker \psi = \ker \phi = \ker(\psi, \phi) = \mu, \tag{20}$$

where μ is the greatest idempotent separating congruence on S.

Proof. From Theorem 8 we know that the cross-connection $[I_S, \Lambda_S; \Gamma_S, \Delta_S]$ is isomorphic to a cross-connection $[L^{\mathrm{op}}, L; \Gamma, \Delta]$, where L is a complemented modular lattice, and where Γ and Δ are given by (9) and (11). Since Γ and Δ are injective, we see that Γ_S and Δ_S are injective. Hence, by Lemma 3.9 of [16] and its dual we then have $\ker \lambda = \ker \rho = \ker(\lambda, \rho)$, where $\ker(\lambda, \rho) = \mu$ follows from the results of [4, 6]. Thus λ, ρ, and (λ, ρ) are equivalent representations.

Obviously $\kappa: L_l \to \Lambda_S$, $Se \to L_e$ is an order isomorphism, and for all $x \in S$, $\kappa \rho_x \kappa^{-1} = \phi_x$. From this we have that $\operatorname{im} \rho \to \operatorname{im} \phi$, $\rho_x \to \phi_x$ is an isomorphism. Analogously, $\operatorname{im} \lambda \to \operatorname{im} \psi$, $\lambda_x \to \psi_x$ is an isomorphism. From this and the above we find $\ker \phi = \ker \rho = \mu = \ker \lambda = \ker \psi$. Whence also $\ker(\psi, \phi) = \ker \psi \cap \ker \phi = \mu$. We conclude that (20) holds, and that all the above considered representations are equivalent.

In the following L will always denote a complemented modular lattice. We shall now investigate the fundamental strongly regular Baer semigroup $B(L)$.

THEOREM 13. *Let $e=(a';a)$ be any idempotent of $B(L)$. Then the semigroup $eB(L)e$ is isomorphic to $B(L(a))$.*

Proof. Consider

$$\theta: B(L(a)) \to B(L), \qquad f \to ef. \tag{21}$$

Clearly ef is a strongly range closed transformation of L and $\operatorname{im} ef \subseteq L(a)$. Since e induces the identity transformation on $L(a)$, we have $efe=ef$, and so $f\theta \in eB(L)e$. If $f\theta = g\theta$, then for all $x \in L(a)$

$$xf=(xe)f=x(ef)=x(f\theta)=x(g\theta)=x(eg)=(xe)g=xg,$$

so that θ is injective.

If $e'=(b';b) \leqslant e$ in the natural partial order on the set of idempotents of $B(L)$, then $b \leqslant a$ and $a' \leqslant b'$ in L, and $b_1 = b' \wedge a$ is a complement of b in $L(a)$. Further,

$$a' \vee b_1 = a' \vee (b' \wedge a) = (a' \wedge a) \vee b' = b'$$

by modularity. For all $x \in L$,

$$\begin{aligned}(x \vee b') \wedge b &= (x \vee a' \vee b_1) \wedge a \wedge b \\ &= (((x \vee a') \wedge a) \vee b_1) \wedge b\end{aligned}$$

by modularity, and thus

$$xe' = xee'',$$

where $e''=(b_1;b)$ is an idempotent in $B(L(a))$. Now if f is any element of $eB(L)e$, then f has a normal factorization which is of the form $f=e'\alpha$, where $e'=(b';b) \leqslant e$ and where α is an isomorphism among principal ideals of L that are contained in $L(a)$. By the above argument $f=ee''\alpha$, where $e''=(b_1;b)$ as before. Since $e''\alpha \in B(L(a))$, it follows that θ is surjective.

Clearly θ is a homomorphism. We have proved that θ is an isomorphism of $B(L(a))$ onto $eB(L)e$.

The group of units of $B(L)$ is exactly $\operatorname{Aut} L$, i.e., the automorphism group of L. We therefore have the following.

COROLLARY 14. *The maximal subgroup H_e of $B(L)$ corresponding to the idempotent $(a';a)=e$ is isomorphic to* $\operatorname{Aut} L(a)$ *(and to* $\operatorname{Aut} L^{\mathrm{op}}(a')$*).*

Recall from [8] that the elements $a, b \in L$ are called algebraically equivalent, that is $a \sim_a b$, if and only if $L(a)$ is isomorphic to $L(b)$. We have the following characterization of Green's relation $\mathscr{D}$ on $B(L)$ in terms of the algebraic equivalence on L.

THEOREM 15. *Let* $f_1, f_2 \in B(L)$, $1f_i = a_i$, $f_i^+ 0 = b_i$, $i = 1, 2$. *Then the following are equivalent.*

(i) $f_1 \mathscr{D} f_2$ *in* $B(L)$,

(ii) $a_1 \sim_a a_2$ *in* L,

(iii) $b_1 \sim_a b_2$ *in* L^{op}.

Proof. Let a_i' be a complement of a_i in L, $i = 1, 2$. Then $e_i = (a_i'; a_i) \mathscr{L} f_i$ in $B(L)$, $i = 1, 2$. Thus $f_1 \mathscr{D} f_2$ in $B(L)$ if and only if $e_1 \mathscr{D} e_2$ in $B(L)$. If $e_1 \mathscr{D} e_2$, then $e_1 B(L) e_1$ is isomorphic to $e_2 B(L) e_2$. Using Theorem 13 we then have

$$B(L(a_1)) \cong e_1 B(L) e_1 \cong e_2 B(L) e_2 \cong B(L(a_2)).$$

Therefore $L(a_1) \cong L(a_2)$ and thus $a_1 \sim_a a_2$. Conversely, if $a_1 \sim_a a_2$, then $L(a_1) \cong L(a_2)$, so that $B(L(a_1)) \cong B(L(a_2))$, and again by Theorem 13, $e_1 B(L) e_1 \cong e_2 B(L) e_2$. From [14] and from the fact that $B(L) \cong T(E(L))$ now follows that $e_1 \mathscr{D} e_2$ in $B(L)$. We have shown that (i) and (ii) are equivalent. Since $f_1 \mathscr{D} f_2$ in $B(L)$ if and only if $f_1^+ \mathscr{D} f_2^+$ in $B(L^{\mathrm{op}})$, the equivalence of (i) and (iii) follows for dual reasons.

Remark 16. If L is a complemented modular lattice and $a, b \in L$, then we say that a and b are *perspective* if a and b have a common complement, and in this case we use the notation $a \not\sim b$. The reflexive symmetric relation $\not\sim$ is called the *perspective relation*, and its transitive closure is an equivalence relation which is called the *projectivity relation*. If $a, b \in L$, then a and b are said to be *projective*, i.e., $a \sim_p b$, if they are related in the projectivity relation. The idempotents $e = (a'; a)$ and $f = (b'; b)$ of $B(L)$ are $\mathscr{D}$-related in $P(L)$ if and only if $a \sim_p b$ in L [17]. Since $P(L)$ is a subsemigroup of $B(L)$ we always have $\not\sim \subseteq \sim_p \subseteq \sim_a$. In case L is coordinatized by a continuous regular ring, then $\not\sim = \sim_p = \sim_a$ (see, e.g. [10]); in this case $f_1 \mathscr{D} f_2$ in $B(L)$ if and only if $1f_1 \not\sim 1f_2$. This property holds for some projective geometries as well (see Theorem 2.27 of [1]).

Let S be a regular semigroup. For $x \in S$, denote the set of inverses of x in S by $V(x)$. Let $\pi(S)$ be the relation on S

$$\pi(S) = \{(x, y) \in S \times S \mid V(x) = V(y)\}. \tag{22}$$

The equivalence relation $\pi(S)$ is in general not a congruence relation. Yet if $\pi(S)$ is a congruence relation, then it is the greatest congruence relation on S whose idempotent congruence classes form rectangular bands. This is, e.g., the case with orthodox semigroups and with V-regular semigroups [12]. In [3, 17, 18] regular semigroups S are considered for which $\pi(S)$ is the identity relation on S. It is easy to see that if $\pi(S)$ is the identity relation on S, then the same holds for every $\mathscr{H}$-coextension of any full regular subsemigroup of S.

THEOREM 17. *If S is a strongly regular Baer semigroup, then $\pi(S)$ is the identity relation on S.*

Proof. Let L be a complemented modular lattice and suppose that $a, b \in L$ have the same set of complements in L. Since L is relatively complemented, $a \wedge b$ has a complement c in $L(b)$. Let $(a \vee b)'$ be a complement of $a \vee b$ in L, and put $d = c \vee (a \vee b)'$. Then

$$\begin{aligned} a \vee d &= a \vee c \vee (a \vee b)' \\ &= a \vee (a \wedge b) \vee c \vee (a \vee b)' \\ &= a \vee b \vee (a \vee b)' = 1, \\ a \wedge d &\leqslant (a \vee b) \wedge d \\ &= (a \vee b) \wedge (c \vee (a \vee b)') \\ &= c \vee ((a \vee b) \wedge (a \vee b)') = c \leqslant b \\ &\qquad \text{(by modularity, and since } c \leqslant a \vee b). \end{aligned}$$

Hence $a \wedge d \leqslant a \wedge b$, $a \wedge d \leqslant c$, and thus $a \wedge d \leqslant (a \wedge b) \wedge c = 0$. Consequently d is a complement of a, and so d is a complement of b also. Hence

$$0 = b \wedge d \geqslant b \wedge c = c$$

from which $c = 0$. This implies $a \wedge b = b$. By symmetry, $a = b$.

Let us now consider the semigroup $B(L)$. Distinct $\mathscr{L}$-classes of $B(L)$ are of the form $L_{(a';a)}$ and $L_{(b';b)}$, where a and b are distinct elements of L, and where a' and b' are complements of a and b, respectively. From the above we can find a complement a'' of a in L that is not a complement of b in L or we can find a complement b'' of b in L that is not a complement of a in L. In the first case $(a''; a) \in E(L_{(a';a)})$ but $R_{(a'';a)} \cap L_{(b';b)}$ contains no idempotent, and in the second case $(b''; b) \in E(L_{(b';b)})$ but $R_{(b'';b)} \cap L_{(a';a)}$ contains no idempotent. Similarly, if $R_{(a;a')}$ and $R_{(b;b')}$ are distinct $\mathscr{R}$-classes of $B(L)$, then there exists an idempotent $(a; a'') \in E(R_{(a;a')})$ such that $L_{(a;a'')} \cap R_{(b;b')}$ contains no idempotent, or there exists an idempotent $(b; b'') \in E(R_{(b;b')})$ such that $L_{(b;b'')} \cap R_{(a;a')}$ contains no idempotent. From Lemma 2.1 of [11] we conclude that $\pi(B(L))$ is the identity relation on $B(L)$.

Let S be any strongly regular Baer semigroup coordinatizing the complemented modular lattice L. Then S is an $\mathscr{H}$-coextension of a full regular subsemigroup of $B(L)$, and consequently $\pi(S)$ is the identity relation on S.

Remark 18. It follows from the foregoing theorem that a strongly regular Baer semigroup S is orthodox if and only if it is an inverse semigroup, and this is the case if and only if the complemented modular lattice $S/\mathscr{L}$ is a Boolean algebra (see also [15] and Remark 11). In particular the multiplicative semigroup of a regular ring R is orthodox if and only if R is an abelian regular ring (see also [15]).

EXAMPLE 19. Let V be a vector space over a division ring D, and let $L = \mathscr{P}(V)$ be the projective geometry which is associated with V. Then the semigroup of all semilinear endomorphisms $S(V)$ can be shown to be a regular semigroup containing the semigroup $LT(V)$ of all linear endomorphisms of V as a full regular subsemigroup [9]. Hence the cross-connection induced by $S(V)$ is isomorphic to $[L^{\mathrm{op}}, L; \Gamma, \Delta]$. By Theorem 8, $S(V)$ is a strongly regular Baer semigroup. Let μ be the greatest idempotent-separating congruence on $S(V)$. Then $S(V)/\mu$ is isomorphic to a full subsemigroup to $B(L)$. It can be shown [9] that in $S(V)$

$$\mu = \{(f, g) \in S(V) \times S(V) \mid g = \alpha f \text{ for some } \alpha \in D\},$$

and that for each $f \in S(V)$ the unique element of $B(L)$ determined by f is the transformation

$$\bar{f}: L \to L, \qquad w \to f(w).$$

Here $\mathfrak{P} : S(V) \to B(L), f \to \bar{f}$ is a canonical homomorphism which induces μ on $S(V)$. In this case $\mathfrak{P}$ is the right Hall representation (or the right Janowitz representation) of $S(V)$. The μ-classes containing idempotents are isomorphic to the group of non-zero elements of D under multiplication. A maximal subgroup of $S(V)$ corresponding to a projection $e(w'; w)$ of V onto $w \in L$ is isomorphic to the group of all semilinear automorphisms of the vector space w, and the subgroup of $B(L)$ corresponding to the idempotent $\mathfrak{P}(e(w'; w)) = (w'; w)$ is the projective group of the geometry $\mathscr{P}(w)$. By the fundamental theorem of projective geometry (see Theorem 2.26 of [1]), all the elements $g \in B(L)$ for which $1g$ is of dimension greater than one belong to the image of $S(V)$ under $\mathfrak{P}$ (see [9]).

When L is a projective geometry, the elements of $B(L)$ are called projective transformations or projective maps. Thus projective maps on L are binormal maps on L. Since the semigroup $B(L)$ uniquely determines and is determined by the geometry L, it is possible to study geometrical properties of L in terms of algebraic properties of the semigroup $B(L)$ or of the semigroup $P(L)$. When L is Desarguian, L is also uniquely determined by the semigroup $S(V)$ of all semilinear endomorphisms, and several results on the group of semilinear isomorphisms of V and the general linear group on V have a natural extension to $S(V)$.

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